NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

British Mathematical Olympiad

1976

Time allowed - 3½ hours

24th March, 1976

Write on one side of the paper only. Start each question on a fresh sheet of paper. Arrange your answers in order.

Put your full name, age (in years and months) and school on the top sheet of your answers. On each other sheet put your name and initials.

In any question, marks may be added for elegance and clarity or subtracted for obscure or poor presentation.

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1. Find, with proof, the length d of the shortest straight line which bisects the area of an arbitrarily given triangle. Express d in terms of the area Δ of the triangle and one of its angles.

   Show that there is a shorter line (not straight) which bisects the area of the given triangle.

2. Prove that if x, y, z are positive real numbers then

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{3}{2}.$$

3. $S_1, S_2, \ldots S_{50}$ are subsets of a finite set E. Each subset contains more than half the elements of E.

   Show that it is possible to find a subset F of E, having not more than 5 elements, such that each $S_i (1 \leq i \leq 50)$ has an element in common with F.

4. Prove that if n is a non-negative integer then $19.8^n + 17$ is not a prime number.

QUESTIONS 5 and 6 ARE OVERLEAF.

5. Prove that

$$\sum_{t=0}^{\frac{1}{2}(r-1)} \binom{n}{r-t}\binom{n}{t}(\alpha\beta)^t\left(\alpha^{r-2t}+\beta^{r-2t}\right) = \sum_{t=0}^{\frac{1}{2}(r-1)} \binom{n}{r-t}\binom{r-t}{t}(\alpha\beta)^t(\alpha+\beta)^{r-2t},$$

where α and β are real numbers, r and n are positive integers with r odd and $r \leqslant n$.

[$\binom{m}{s}$ denotes the coefficient of x^s in the expansion of $(1 + x)^m$.]

6. A sphere with centre O and radius r is cut in a circle K by a horizontal plane distant $\frac{1}{2}r$ above O. The part of the sphere above the plane is removed and replaced by a right circular cone having K as its base and having its vertex V at a distance $2r$ vertically above O.

Q is a point on the sphere on the same horizontal level as O. The plane OVQ cuts the circle K in two points X and Y, of which Y is the further from Q. P is a point of the cone lying on VY, whose position can be determined by the fact that the shortest path from P to Q over the surfaces of cone and sphere cuts the circle K at an angle of 45°.

Prove $VP = \sqrt{3}r/\sqrt{(1+1/\sqrt{5})}$.

[In a spherical triangle ABC the sides are arcs of great circles (centre O) and the sides are measured by the angles they subtend at O. You <u>may</u> find these spherical triangle formulae useful:

$$\sin a/\sin A = \sin b/\sin B = \sin c/\sin C$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A].$$